1901001101010001
EXAMINATION FEBRUARY-MARCH 2024
MASTER OF ARTS PART-I (EXTERNAL)
MATHEMATICS
MEASURE THEORY - LEVEL 1

[Time: As Per Schedule] **[Max. Marks: 100]**

Instructions:

1. Fill up strictly the following details on your answer book
   - a. Name of the Examination: **MASTER OF ARTS PART-1 (EXTERNAL)**
   - b. Name of the Subject: **MATHEMATICS MEASURE THEORY – LEVEL 1**
   - c. Subject Code No: **1901001101010001**

2. Sketch neat and labelled diagram wherever necessary.
3. Figures to the right indicate full marks of the question.
4. All questions are compulsory.
5. Attempt any FIVE questions of the following.
6. Each question carries equal marks.
7. Follow usual notations and conventions.

Seat No:

Student’s Signature

Q.1 (a) Let f and g be two nonnegative measurable functions. If f is integrable over E and $g(x) < f(x)$ on E, then prove that g is also integrable over E and **7**

$$\int_E f - g = \int_E f - \int_E g.$$

(b) Let A be any set and $E_1, \ldots E_n$ be a finite sequence of disjoint measurable sets. Then prove that $m^*(A \cap [\bigcup_{i=1}^{n} E_i]) = \sum_{i=1}^{n} m^*(A \cap E_i)$. **7**

(c) Define upper and lower Riemann integral of a function. **6**

If $f(x) = \begin{cases} 0; x \text{ is rational} \\ 1; x \text{ is irrational} \end{cases}$

then show that $R\int_a^{\bar{b}} f(x)dx = b - a$ and $R\int_{\underline{a}}^{b} f(x)dx = 0$.

OR

(a) State and prove Fatau's lemma. **7**

(b) Let $\langle A_n \rangle$ be a countable collection of sets of real numbers. Then prove that **7**

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty} m^* A_n.$$

(c) Let f be nonnegative function then show that $\int_E f = 0 \Rightarrow f = 0$ almost everywhere. **6**

Q.2 (a) If two functions are absolutely continuous then prove that their sum and product are also absolutely continuous. **7**

(b) If f and g are bounded measurable functions defined on a set E of finite measure, then prove that **7**

(i) $\int_E (af + bg) = a\int_E f + b\int_E g$

(ii) If $f = g$ almost everywhere then $\int_E f = \int_E g$

(iii) If $A \le f(x) \le B$ then $AmE \le \int_E f \le BmE$.

(c) Define bounded variation. If $f \in BV([a,b])$, then prove that $T_a^b = P_a^b + N_a^b$ and $f(b) - f(a) = P_a^b - N_a^b$. **6**

OR

(a) Prove that every singleton set $\{b\}, 0 < b \le 1$, is a Borel set. **7**

(b) Let f be define and bounded on a measurable set E with mE finite. In order that $\underset{f\le\psi}{\inf}\int_E \psi(x)dx = \underset{f\ge\phi}{\sup}\int_E \phi(x)dx$, for all simple function ϕ and ψ, show that it is necessary and sufficient that f be measurable. **7**

(c) (i) Show that $m^*E = 0$ then E is a measurable set. **6**

(ii) Prove that a constant function defined over a measurable set is measurable.

Q.3 (a) Let $c \in (a,b)$. If $f \in BV([a,c])$ and $f \in BV([c,b])$ then prove that $f \in BV([a,b])$ and moreover $T_a^b(f) = T_a^c(f) + T_c^b(f)$. **7**

(b) Let $\langle f_n \rangle$ be a sequence of measurable functions and let g be integrable over E such that $|f_n| \le g$ on E and $f(x) = \lim_{n\to\infty} f_n(x)$ then prove that **7**

$$\int_E f = \lim_{n\to\infty}\int_E f_n.$$

(c) If f is integrable on $[a,b]$ and if $\int_a^x f(t)dt = 0, \forall x \in [a,b]$ then prove that $f(t) = 0$ almost everywhere on $[a,b]$. **6**

OR

(a) If ϕ is convex on (a,b) and if x, y, x', y' are points of (a,b) with $x \leq x' < y'$ and $x < y \leq y'$, then prove that the chord over (x', y') has larger slope than the chord over (x, y); that is $\frac{\phi(y)-\phi(x)}{y-x} \leq \frac{\phi(y')-\phi(x')}{y'-x'}$. **7**

(b) State and prove monotone convergence theorem. **7**

(c) Prove that f is integrable if and only if $|f|$ is integrable. **6**

Q.4 (a) Define a continuous operator. If $T: D(T) \to Y$ is a linear operator, where $D(T) \subset X$ and X, Y are the normed spaces, then prove that T is continuous if and only if T is bounded. **7**

(b) Let $\{x_1, x_2, .., x_n\}$ be linearly independent set of vectors in a normed space X. Then there exists $C > 0$ such that for every choice of scalars $\alpha_1, \alpha_2, \dots, \alpha_n$; prove that **7**

$$\left\|\sum_{i=1}^{n} \alpha_i x_i\right\| \geq C \sum_{i=1}^{n} |\alpha_i| .$$

(c) Prove that finite dimensional vector space is algebraically reflexive. **6**

OR

(a) Prove that the space $C[a,b]$ is complete. **7**

(b) Define discrete metric and show that it satisfies all the four properties of a metric space. **7**

(c) Let $T: X \to Y$ and $S: Y \to Z$ be bijective linear operators, where X, Y, Z are vector spaces. Then prove that $(ST)^{-1}: Z \to X \ of \ (ST)$ exists, and $(ST)^{-1} = T^{-1}S^{-1}$. **6**

Q.5 (a) Let X be an inner product space and let M be a complete convex subspace Y. Let $x \in X$ be fixed, then prove that $z = x - y$ is orthogonal to Y. **7**

(b) Show that the dual space of l^p is l^q; here $1 < p < +\infty$ and q is the conjugate of p, that is $1/p + 1/q = 1$. **7**

(c) Let X be an inner product space. Then prove that the corresponding norm satisfies $|\langle x, y \rangle| \leq ||x|| \, ||y||$, where the equality holds if and only if $\{x, y\}$ is a linearly dependent set. **6**

OR

(a) Show that in an inner product space, if $x \perp y$ then $||x + y||^2 = ||x||^2 + ||y||^2$. Does the converse hold true? Justify. **7**

(b) Let $T: X \to Y$ be a linear operator, where X and Y are vector space then prove that **7**

(i) The range $R(T)$ is a vector space.

(ii) If dim $D(T) = n < \infty$, then dim $R(T) \leq n$.

(c) Let M be a subset of a Hilbert space H then prove that **6**

(i) $M \cap M^{\perp} = \{0\}$

(ii) $M \subset M^{\perp\perp}$.

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